1a) Let $I = (a, b]$, where $a < b$. Show that the set of cluster point of I is $[a, b]$.

b) Write down the set of cluster points of following subsets of $\mathbb{R}$:

(i) $\{x_1, x_2, \ldots, x_N\}$

(ii) $\mathbb{N}$

(iii) $\{\frac{1}{n} : n \in \mathbb{N}\}$

(iv) $I \cap \mathbb{Q}$, where $\mathbb{Q}$ is rational number, $I = [0, 1]$

2. Use the definition of limit to show that

$$\lim_{x \to -1} \frac{x^2 + 2x + 4}{x + 2} = 3$$

3 Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Suppose c is cluster point of A.

(a) Prove the $\boxed{\text{Sequential Criterion for limit}}$: (i) $\lim_{x \to c} f(x) = l$ $(\in \mathbb{R})$ if and only if (ii) every sequence (x_n) in $A \setminus \{c\}$ converging to c, $(f(x_n))$ converges to l.

(b) Suppose $\lim_{x \to c} f(x)$ does not exist. Show that there exists $\varepsilon_0 > 0$ and two sequences (x_n) and (y_n) in $A \setminus \{c\}$, both converging to c, such that $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

(c) Prove the $\boxed{\text{Cauchy Criterion for limit}}$: (i) $\lim_{x \to c} f(x)$ exists iff (ii) for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in A$ with $0 < |x - c|, |y - c| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

(4) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

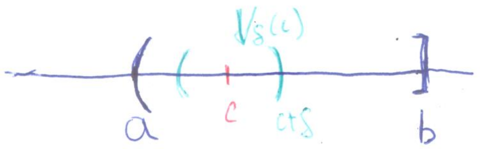
$$f(x) := \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

(a) Show that f has a limit at $x = 0$

(b) Show that if $c \neq 0$, then f does not have a limit at c.

1 a) Definition : Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a cluster point of A if for every $\delta > 0$, there exists at least one point $x \in A$, $x \neq c$ such that $|x - c| < \delta$

$\Leftrightarrow$ every δ-neighborhood $V_\delta(c) = (c-\delta, c+\delta)$ of c contains at least one point of A distinct from c



Let $\delta > 0$. if $a < c < b$, $\exists x_1 \in (a, b)$, $a < x_1 < c$

if $|c - x_1| < \delta$, take $x = x_1$, done.

if $|c - x_1| \geq \delta$, take $x = c - \frac{\delta}{2}$,

$\because$ $c - x_1 \geq \delta \Rightarrow c - \delta \geq x_1 > a$

$\therefore$ $x > a$ and $|x - c| = \frac{\delta}{2} < \delta$

Hence all points in (a, b) are cluster point of I

if $c = a$, take $x = c + \frac{\delta}{2} = a + \frac{\delta}{2}$

if $c = b$, take $x = c - \frac{\delta}{2} = b - \frac{\delta}{2}$

if $c > b$, choose $\delta = \frac{c-b}{2}$, $V_\delta(c) = \left(\frac{c+b}{2}, c + \frac{c-b}{2}\right)$

$\because$ $\frac{c+b}{2} > b$, $V_\delta(c) \cap A = \phi$

Similarly if $c < a$, $\exists \delta$ s.t. $V_\delta(c) \cap A = \phi$

Hence if $a > c$ or $c > b$, c is not cluster point of I

Then cluster point of I = $[a, b]$.

1\. b (i) ϕ

(ii) ϕ

(iii) $\{0\}$, since there is a sequence with limit tends to 0.

(iv) $[0,1]$, since $\mathbb{Q}$ is dense in $\mathbb{R}$.

$A = \mathbb{R}$

2\. $f(x) = \dfrac{x^2+2x+4}{x+2}$, to show $\lim\limits_{x\to -1} f(x) = 3$

By definition, to show, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

if $x \in A$ and $0 < |x+1| < \delta$,

then $|f(x) - 3| < \varepsilon$.

$$|f(x) - 3| = \left|\frac{x^2+2x+4}{x+2} - 3\right|$$

$$= \left|\frac{x^2 - x - 2}{x+2}\right|$$

$$= \left|\frac{(x-2)(x+1)}{x+2}\right|$$

$$= \left|\frac{x-2}{x+2}\right| |x+1|$$

Get the upper bound of $\left|\frac{x-2}{x+2}\right| = \left|1 - \frac{4}{x+2}\right| \le 1 + \left|\frac{4}{x+2}\right|$

if $|x+1| < \frac{1}{2}$, $-\frac{3}{2} < x < -\frac{1}{2}$

$\Rightarrow \frac{1}{2} < x+2 < \frac{3}{2}$

$\Rightarrow \frac{1}{2} < |x+2| < \frac{3}{2}$

$\Rightarrow \frac{2}{3} > \frac{1}{|x+2|} > \frac{1}{2}$

Then $\left|\frac{x-2}{x+2}\right| < 1 + 4 \cdot \frac{2}{3} = \frac{11}{3}$

Choose $\delta = \inf\{\frac{3}{11}\varepsilon, \frac{1}{2}\}$, if $0 < |x+1| < \delta$, $|f(x) - 3| < \frac{11}{3}|x+1|$

$< \varepsilon$.

Then $\lim\limits_{x\to -1} f(x) = 3$.

3 a) (i) $\Rightarrow$ (ii) Assume $\lim_{x\to c} f(x) = \ell$. and (x_n) in $A\setminus\{c\}$ sequence converging to c

Want to Show $\lim f(x_n) = \ell$.

let $\varepsilon > 0$, $\because \lim_{x\to c} f(x) = \ell$

$\exists\, \delta > 0$ s.t. if $x \in A$ and $0 < |x-c| < \delta$, then $|f(x) - \ell| < \varepsilon$

$\because \lim x_n = c$, for above δ; $\exists\, K(\delta) \in \mathbb{N}$ s.t. if $n \geq K(\delta)$

$|x_n - c| < \delta$,

$\because x_n \neq c$, $0 < |x_n - c| < \delta$

Choose above K, if $n \geq K$, $0 < |x_n - c| < \delta$

$\Rightarrow |f(x_n) - \ell| < \varepsilon$

(ii) $\Rightarrow$ (i) : Suppose (i) is not true

$\lim_{x\to c} f(x) \not\to \ell$

$\exists\, \varepsilon_0 > 0$ s.t. $\forall\, \delta > 0$. $\exists\, x_\delta \in A$, $0 < |x_\delta - c| < \delta$ s.t. $|f(x) - \ell| \geq \varepsilon_0$

Choose $\delta = \frac{1}{n}$, $\exists\, x_n \in A$ s.t. $0 < |x_n - c| < \frac{1}{n}$

but $|f(x_n) - \ell| \geq \varepsilon_0$

That is, $\lim_n x_n = c$, $f(x_n) \not\to \ell$

Contradict to (ii)

Then (ii) $\Rightarrow$ (i)

f(x) does not have limit L at c

b) Form (a), we know $\underline{\lim_{x\to c} f(x) \neq \ell}$

$\Leftrightarrow \exists\, (x_n)$ in $A\setminus\{c\}$ sequence s.t. $\lim_n x_n = c$, but $\underline{\lim_n f(x_n) \neq \ell}$

sequence $(f(x_n))$ does not converge to ℓ

Then, we have f does not have limit at c

$\Leftrightarrow \exists$ sequence (x_n) in $A\setminus\{c\}$ s.t. $\lim_n x_n = c$, but sequence $(f(x_n))$ does not converge.

3(b) " Since $(f(x_n))$ does not converge

By Cauchy Criterion of sequence

$\exists\, \varepsilon_0 > 0$, $\exists\, (y_n), (z_n)$ subsequence of (x_n), then $\lim y_n = \lim z_n$
$= \lim x_n = c$

and $|f(y_n) - f(z_n)| \geq \varepsilon_0 \quad \forall n \in \mathbb{N}$

(c) (i) $\Rightarrow$ (ii): Suppose $\lim_{x\to c} f(x)$ exists

i.e. $\lim_{x \to c} f(x) = l$ for some $l \in \mathbb{R}$

Let $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. if $x \in A \setminus \{c\}$ with $|x - c| < \delta$,

then $|f(x) - l| < \frac{\varepsilon}{2}$

if $x, y \in A \setminus \{c\}$ with $|x-c|, |y-c| < \delta$

Then $|f(x) - l| < \frac{\varepsilon}{2}$

$|f(y) - l| < \frac{\varepsilon}{2}$

$\Rightarrow |f(x) - f(y)| \leq |f(x) - l| + |f(y) - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$

(ii) $\Rightarrow$ (i): Suppose (i) is not true and (ii) is true

By 3(b), $\exists\, \varepsilon_0 > 0$, $\exists\, (x_n), (y_n)$ sequence in $A \setminus \{c\}$

$\lim x_n = \lim y_n = c$, and $|f(x_n) - f(y_n)| \geq \varepsilon_0$

By (ii), for this ε_0, $\exists\, \delta > 0$ s.t. if $x, y \in A \setminus \{c\}$, $|x-c|, |y-c| < \delta$,

then $|f(x) - f(y)| < \varepsilon_0$

Since $\lim x_n = \lim y_n = c$, $\exists\, N \in \mathbb{N}$ s.t. $|x_n - c|, |y_n - c| < \delta$ if $n \geq N$

Then by above $|f(x_n) - f(y_n)| < \varepsilon_0$

contradiction.

4 a) Let $\varepsilon > 0$, WTS $|f(x) - 0| < \varepsilon$, for some δ

$$|f(x) - 0| = \begin{cases} |x| & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Choose $\delta = \varepsilon$, then if $0 < |x - 0| < \delta = \varepsilon$

Then $|f(x) - 0| < |x| < \delta = \varepsilon$.

Then $\lim_{x \to 0} f(x) = 0$

b) By 3(c), we need to find two sequence (x_n), (y_n) in $A \setminus \{c\}$

with $\lim x_n = \lim y_n = c$, but $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for some $\varepsilon_0 > 0$

since $\mathbb{Q}$ is dense in $\mathbb{R}$ $\qquad$ $\lim_n f(x_n) \neq \lim_n f(y_n)$

$\exists (x_n) \in \mathbb{Q}$ sequence s.t. $\lim_n x_n = c$

$\lim f(x_n) = \lim x_n = c$

Since irrational number is dense in $\mathbb{R}$

$\exists (y_n)$ sequence of irrational number s.t. $\lim_n y_n = c$

$\lim_n f(y_n) = \lim_n 0 = 0$

$\therefore \lim_n f(x_n) \neq \lim_n f(y_n)$

$\lim_{x \to c} f(x)$ does not exist.